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# EVOLUTION OF HOMOGENEOUS TURBULENCE IN A DENSITY-STRATIFIED MEDIUM. 2. ASYMPTOTIC ANALYSIS OF FINAL STAGE OF DECAY

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*The asymptotics of a large evolution time of homogeneous turbulence in a density-stratified medium is studied. It is shown that the presence of stratification substantially changes the attenuation rate of turbulence kinetic energy and other turbulence characteristics and forms asymptotic regimes which differ from the case of an isotropic medium. The effect of the molecular Prandtl number on the final stage of turbulence decay is studied in detail.*

We consider the evolution of turbulence generated by a turbulizing grid in a medium with a uniform velocity field in the presence of a constant transverse density gradient caused by a gravity force field.

Modeling of the evolution of homogeneous turbulence in a density-stratified medium is one of basic test problems on which modern turbulent models of a scalar field are studied and verified. On the other hand, this problem has a certain practical value for studying turbulent transfer in ocean and atmospheric flows where gravity-induced transverse stratification usually exists.

We analytically studied a model of homogeneous turbulence in a density-stratified medium [1] that was a particular case of the more common second-order model of moments [2]. In earlier works [1] the model was studied in detail. It is shown, in particular, that this model describes the generation and propagation of internal turbulent gravitation waves, as a result of which at the final stage of decay the characteristics of turbulent velocity and density fields substantially differ from the case of an isotropic medium.

The study of an asymptotic regime of turbulence decay for $\tau \to \infty$ provides information about whether the field of turbulent oscillations in a stratified medium should be plane at the final stage of decay, what portion of the turbulent energy is contained in the internal gravitation waves, and how a wave interacts with small-scale random oscillations (see [1]).

In a previous paper [3] the evolution of turbulence at large $\tau$ was considered analytically by the small-parameter method. Mathematical systems which corresponded to fluctuations and fluctuation-averaged quantities were distinguished and analytical relations for the frequency and amplitude of the internal turbulent gravitation wave were obtained. In what follows we study the final stage of the decay of turbulence evolution, which can be treated as the far-field limit considered earlier [3]. This study should provide values of the model functions that are limiting for $\tau \to \infty$, the rate of convergence on these asymptotic values, and the dependence of this rate on the Prandtl number.

In [3] the system of differential equations of the model of [1] is reduced to the form

$$t\frac{dK}{d\tau} = \varepsilon\frac{-7d'(K - 1/3)}{R} + 2\varepsilon q(K - 4/5),$$

$$t\frac{dR}{d\tau} = \varepsilon\frac{4}{5}d(1 - R/R_\infty) - 2\varepsilon q(1 - \alpha_2 R)R,$$

$$t\frac{d\vartheta}{d\tau} = 2\varepsilon\left(\frac{1}{R} - 1\right)\vartheta + 2\varepsilon q(1 + \vartheta),$$

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$$t\frac{dq}{d\tau} = t^2 A_1 + \varepsilon q\left[\frac{1}{R}\left(2 - \frac{d'}{3}\right) - 10d'K - \alpha_1 + p\right] + 2\varepsilon q^2,$$

$$\frac{dt}{d\tau} = \varepsilon p, \quad t\frac{dE}{d\tau} = -\frac{2\varepsilon E}{R} - 2\varepsilon qE, \tag{1}$$

where $d = d(R_\lambda^2) = 1 - 2/(1 + \sqrt{1 + \delta_u/R_\lambda^2})$ is the parameter of different scales in [2]; $d \in (0,1)$; $d' = 1 - d$; $\delta_u \approx 2800$; $\sigma_\infty$ is the asymptotic value of the turbulent Prandtl number when $\tau \to \infty$ and $Fr = 0$; $R_\infty$ is the asymptotic value of the ratio of scales $T_u/T_p$ when $\tau \to \infty$ and $Fr = 0$; $\alpha_1 = 2d(\sigma_\infty + 3/5)/R_\infty$; $\alpha_2 = (\alpha/(\sigma + 1))\alpha_1$; $p = 4d/5R_\infty + 5d'/3R > 0$ and

$$A_1 = K + \vartheta\,[d\,(K - 1/3) - 2/3]. \tag{2}$$

The asymptotic values of $\sigma_\infty$ and $R_\infty$ are taken from [4]

$$\alpha_\infty = \frac{3\,(1 - \alpha)}{10\sigma}\left[1 - \left(\frac{2\sigma}{1 + \sigma}\right)^{3/2}\right]^{-1}, \tag{3}$$

$$R_\infty = \frac{1}{5\sigma}\left[1 - \left(\frac{2\sigma}{1 + \sigma}\right)^{3/2} + \sigma^{3/2}\right]\left[1 - 2\left(\frac{2\sigma}{1 + \sigma}\right)^{1/2} + \sigma^{1/2}\right]^{-1}. \tag{4}$$

Since the kinetic energy of turbulence $E$ enters into the first five equations of system (1) only in terms of the parameter $d$, it is convenient for analysis to consider the equation for $d(R_\lambda^2)$, which is easily obtained from (1):

$$t\frac{d\,(d)}{d\tau} = \varepsilon p_1\left[\frac{F_u^{**} - 4}{R} - 2\,(2 - \alpha_2 R)\,q\right], \tag{5}$$

where $p_1 = -dd'/(1 + d)$ is the logarithmic derivative of $d$ with respect to $R_\lambda^2$.

We consider the final stage of turbulence decay for $\tau \to \infty$ and $R_\lambda << 1$. The equation for $q$ degenerates in the far field to an algebraic one $A_1 = 0$, which is fulfilled with a high degree of accuracy. In the vicinity of asymptotically weak turbulence the parameter $d(R_\lambda)$ is close to unity. When $d \to 1$, $t >> 1$, system of equations (1) and (5) can be written in the form

$$\frac{4}{5}\frac{t}{R_\infty}\frac{dK}{dt} = -7\frac{d'(K - 1/3)}{R} + 2q\,(K - 4/5),$$

$$\frac{4}{5}\frac{t}{R_\infty}\frac{dR}{dt} = \frac{4}{5}(1 - R/R_\infty) - 2\,(1 - \alpha_2 R)\,Rq,$$

$$K + \vartheta\,(d\,(K - 1/3) - 2/3) = 0, \tag{6}$$

$$\frac{4}{5}\frac{t}{R_\infty}\frac{d\vartheta}{dt} = 2\,(R^{-1} - 1)\,\vartheta + 2q\,(1 + \vartheta),$$

$$\frac{4}{5}\frac{t}{R_\infty}\frac{d\,(d)}{dt} = -\frac{d'}{2}\left[-\frac{6}{5}R^{-1} - 2\,(2 - \alpha_2 R)\,q\right].$$

We present the solutions of system of equations (6) as the sum of their asymptotic values when $\tau \to \infty$ (we designate them by subscript $A$) and of additions tending to zero when $t \to \infty$ (we designate them by the same letter with a prime). We seek these additions in the form of functions of $t$ that decay exponentially:

$$f = f_A + f', \quad f' = C_i t^{-\beta_i}, \tag{7}$$

where the subscript $i$ takes values of $K$, $R$, $\vartheta$, $q$, $d$, and $f$ is one of the functions of $K$, $R$, $\vartheta$, $q$, $d$. In order that the function $f'$ in (7) can decay, it is necessary that all exponents $\beta_i$ be positive.

As $K$, $R$, $\vartheta$, $q$, $d$ tend to their asymptotic values, the derivatives on the left-hand sides of system (6) tend to zero. Thus, the asymptotic values satisfy the system of algebraic equations

$$0 = 2q_A (K_A - 4/5), \quad 0 = \frac{4}{5}(1 - R_A/R_\infty) - 2(1 - \alpha_2 R_A) R_A q_A,$$
$$0 = 2\left(\frac{1}{R_A} - 1\right)\vartheta_A + 2q_A(1 + \vartheta_A), \quad K_A + \vartheta_A(K_A - 1) = 0. \tag{8}$$

It is obvious that equality to zero in the first line of (8) is possible in two cases: A) at $q_A = 0$ and B) when $q_A \neq 0$. In the latter case, $K_A = 4/5$. We also distinguish the subcases A1) $q_A = 0$, $\vartheta \neq 0$ and A2) $q_A = 0$, $\vartheta_A = 0$.

In case A1 we find from system (8) successively (from Eq. (8), in parentheses above the equal sign)

$$q_A \overset{(1)}{=} 0, \quad R_A \overset{(2)}{=} R_\infty, \quad \vartheta_A \overset{(4)}{=} K_A/(1 - K_A), \quad R_A \overset{(3)}{=} 1, \quad R_\infty \overset{(2),(3)}{=} 1. \tag{9}$$

The value of $K_A$ remains undetermined, since at $R_A = R_\infty = 1$ the pair from the second and third equations in (8) is degenerate. The asymptotic value of $R_\infty$, which is equal to unity, and, consequently, case A1 correspond to the molecular Prandtl number $\sigma = 1$.

In case A2 we obtain, respectively

$$q_A \overset{(1)}{=} 0, \quad R_A \overset{(2)}{=} R_\infty, \quad K_A \overset{(4)}{=} 0, \quad \vartheta_A = 0, \tag{10}$$

and in case B

$$K_A \overset{(1)}{=} \frac{4}{5}, \quad \vartheta_A \overset{(4)}{=} 4, \quad q_A \overset{(3)}{=} \frac{4}{5}(1 - R_A^{-1}), \tag{11}$$

where $R_A$ is determined as a root of the quadratic equation

$$\alpha_2 R_A^2 - \left(1 + \alpha_2 + \frac{5}{8}\alpha_3\right) R_A + \frac{3}{2} = 0, \tag{12}$$

and $\alpha_3 = 4d/5R_\infty$. The selection of the root of (12) and the ranges of $\sigma$ to which the asymptotic values from (10) and (11) correspond will be considered later.

We introduce equations for the above-mentioned additions, i.e., the primed functions. For this purpose we subtract term-by-term relations (8) from (6). Then it is necessary to linearize this system of equations, discarding the quadratic terms. Depending on the considered case, A1, A2 or B, we obtain different systems.

For case A1 after linearization

$$\frac{4}{5} t \frac{dK'}{dt} = -7d'(K_A - 1/3) + 2q'(K_A - 4/5), \quad \frac{4}{5} t \frac{dR'}{dt} = \frac{4}{5}(-R'),$$
$$\frac{4}{5} t \frac{d\vartheta'}{dt} = -2\vartheta_A R' + 2q'(1 + \vartheta_A),$$
$$(1 + \vartheta_A) K' + \vartheta'(K_A - 1) - d'\vartheta_A(K_A - 1/3) = 0,$$
$$\frac{4}{5} t \frac{d(d')}{dt} = -(3/5) d'. \tag{13}$$

For case A2

$$1)\ \frac{4}{5}\frac{t}{R_\infty}\frac{dK'}{dt}=\frac{7}{3}\frac{d'}{R_\infty}-\frac{8}{5}q',$$

$$2)\ \frac{4}{5}\frac{t}{R_\infty}\frac{dR'}{dt}=-\frac{4}{5}R'/R_\infty-2\,(1-\alpha_2R_\infty)\,R_\infty q',\tag{14}$$

$$3)\ \frac{4}{5}\frac{t}{R_\infty}\frac{d\vartheta'}{dt}=2\left(\frac{1}{R_\infty}-1\right)\vartheta'+2q',\quad 4)\ K'-\vartheta'=0,\quad 5)\ \frac{4}{5}\frac{t}{R_\infty}\frac{d(d')}{dt}=-\frac{3d'}{5R_\infty}.$$

And, finally, in case B the system for additions will have the form:

$$\frac{4}{5}\frac{t}{R_\infty}\frac{dK'}{dt}=-\frac{49}{15}d'+\frac{8}{5}(1-R_A^{-1}),$$

$$\frac{4}{5}\frac{t}{R_\infty}\frac{dR'}{dt}=-\frac{4}{5}(1-R_A/R_\infty)\,d'-\frac{4}{5}(R_\infty^{-1}+2\,(1-R_A^{-1})\,(1-2\alpha_2R_A))\,R_A',$$

$$-2\,(1-\alpha_2R_A)\,R_Aq',\tag{15}$$

$$\frac{4}{5}\frac{t}{R_\infty}\frac{d\vartheta'}{dt}=\frac{2}{5}\left(\frac{1}{R_A}-1\right)\vartheta'+10q'-\frac{8}{R_A^2}R',$$

$$5K'-\vartheta'/5-\frac{28}{11}d'=0,$$

$$\frac{4}{5}\frac{t}{R_\infty}\frac{d(d')}{dt}=\frac{d'}{2}\left\{-\frac{6}{5R_A}-2\,(2-\alpha_2R_A)\,\frac{4}{5}(1-R_A^{-1})\right\}.$$

We consider alternately cases A1, A2, and B, which are described by systems (13)-(15). When solving these systems one should allow for the fact that the equations entering into them are valid with the accuracy up to discarded terms of a higher order of smallness. The discarded terms are trivally small when $t\to\infty$. However, depending on the values of the exponents $\beta_i$, the auxiliary terms in these systems can become small. This fact should be taken into account in asymptotic analysis. We carry out this analysis using symbols of the degree of convergence on an asymptotic value, which are defined as $O(f')=d\,(\ln f')/d\ln t$.

In case A2, functions $K'$ and $\vartheta'$ are equivalent. Therefore, we exclude $\vartheta'$ from consideration. It follows from the fifth equation of (14) that $O(d')=-3/4$. The second equation in (14) is separated from the system. We find from the third equation of (14) that $O(K')\ge O(q')$, otherwise the function $\vartheta'$ disappears from the system and the system becomes contradictory. When the strict inequality $O(K')>O(q')$ is fulfilled, in analyzing the first equation one should recognize that $O(K')=O(d')=-3/4$; however, this order of $K'$ decrease does not correspond to that in the third equation, since $O(K')=O(\vartheta')=5/2(R_\infty-1)$. Consequently, we conclude that $O(K')=O(q')$. The order of $O(d')$ smallness in the first equation can be smaller than the order of $q'$ or equal to it, $O(d')\le O(q')$. We consider these versions separately.

*Case A2a:* $O(K')=O(\vartheta')=O(q')=O(d')=-3/4$.

Substituting the assumed solutions $\vartheta'=K'=c_Kt^{-3/4}$, $q'=c_qt^{-3/4}$, $d'=c_dt^{-3/4}$ into the first and third equations of (14), we determine the ratio of the coefficients $c_K/c_q$ and $c_d/c_q$

$$c_K/c_q=c_\vartheta/c_q=\left(1-\frac{13}{10R_\infty}\right)^{-1},\quad c_d/c_q=-\frac{9}{35}c_K/c_q+\frac{24}{35}R_\infty.\tag{16}$$

It follows from the second equation of (14) that $O(R') = O(q')$, since, otherwise $(O(R') > O(q'))$, the exponent of $O(R') = -1$ from the first equation of (14) turns to be smaller than that of $O(R') = -3/4$ and the assumption is inconsistent. We find the ratio $c_R/c_q$ from (14)

$$c_R/c_q = 10R_\infty^2 (1 - \alpha_2 R_\infty) . \tag{17}$$

The coefficient $c_q$ remains undetermined.

*Case A2b:* $O(K') = O(\vartheta') > O(d')$.

When $t \to \infty$ the function $d'$ degenerates to zero more rapidly than the other functions. Replacing $\vartheta'$ by $K'$ in the third equation of (14) and eliminating $q'$ from (14-1) and (14-3), we obtain

$$O(K') = -\frac{10}{9}(R_\infty - 1) . \tag{18}$$

The degree of convergence on zero for the additions is determined by the first equation of (14) and is equal to

$$\beta_1 = \frac{10}{9}(R_\infty - 1) . \tag{19}$$

From Eqs. (14-1) and (14-3) we obtain the relation

$$q' = \frac{5}{9}(1 - R_\infty) K' . \tag{20}$$

Substituting the assumed exponential relations with the exponent $-\beta_i$ into system of equations (14), we find the ratios $c_K/c_q$, $c_R/c_q$:

$$c_K/c_q = \frac{2}{\beta_1} R_\infty , \quad c_R/c_q = -\frac{5}{2} \frac{(1 - \alpha_2 R_\infty) R_\infty}{(1 - \beta_1)} . \tag{21}$$

We complete the consideration of case A2 by comparing the expressions for two different versions: A2a and A2b. The exponent $\beta_1 > 0$; therefore, we conclude from (19) that $R_\infty > 1$. On the other hand, $\beta_1 < \beta_2 = 3/4$; otherwise, discarding of the term with $d'$ in (14-1) at large $t$ will be unjustified. At small $\sigma$, the exponent $\beta_1 > 3/4$ and version A2a is realized; starting from some threshold value of $\sigma_{01}$, which is determined by the relation

$$-\frac{10}{9}(R_\infty(\sigma_{01}) - 1) = -\frac{3}{4} ,$$

version A2b is realized.

Cases B and A1 are analyzed by approximately the same scheme. Omitting details we note that, as in case A2, two versions are possible for case B, which will be called B1 and B2. In version B1

$$O(K') = O(\vartheta') = O(R') = O(q') > O(d') \tag{22}$$

and in version B2

$$O(K') = O(\vartheta') = O(R') = O(q') = O(d') . \tag{23}$$

In case B1 the degree of convergence on zero of the corrections is determined by the first equation of (15), since the function $d'$ disappears from it. Designating $O(K') = O(\vartheta') = O(R') = O(q') = -\beta$, we find that

$$\beta_1 = -2R_\infty (1 - 1/R_A) . \tag{24}$$

The asymptotic value of the ratio of time scales of the velocity field and the scalar field $R_A$ is determined for case B from the solution of quadratic equation (12). According to the Vieta theorem, both roots of this equation are positive. We write the solution of Eq. (12) at $d = 1$

$$R_A = \frac{1}{2\alpha_2} + \frac{1}{2} + \frac{1}{4R_\infty \alpha_2} \pm \frac{\sqrt{(1 + \alpha_2 + 1/2R_\infty)^2 - 6\alpha_2}}{2\alpha_2} \tag{25}$$

and require from its physical meaning that its discriminant $D > 0$. We transform $D$

$$D = (1 - \alpha_2 + 1/2R_\infty)^2 + 2\alpha_2 (1/R_\infty - 1).$$

Thus, $D > 0$ if $R_\infty < 1$. This fact indicates that case B refers to the values $R_\infty(\sigma) < 1$ and, consequently, to molecular Prandtl numbers $\sigma$ higher than unity.

For $\beta_1 > 0$ it is necessary that $R_A - 1 < 1$ in (22). It follows from equality (25) that

$$2\alpha_2 (R_A - 1) = (1 - \alpha_2 + 1/2R_\infty) \pm \sqrt{(1 - \alpha_2 + 1/2R_\infty)^2 + 2\alpha_2 (1/R_\infty - 1)} < 0,$$

from which it is seen that the root in (25) should be taken with the minus sign. Thus, the criterion for selection of one of the roots of quadratic equation (12) is obtained for branch B1.

For the second branch, B2, the exponent is equal for all functions: $O(K') = O(\vartheta') = O(R') = O(q') = O(d') = -\beta_2$, with the exponent $\beta_2$ being determined in this case from the equation for $d'$ of system (15), i.e., when $d \to 1$, from

$$-\frac{4}{5R_\infty}\beta_2 = -\frac{3}{5R_A} - (2 - \alpha_2 R_A)\frac{4}{5}(1 - 1/R_A),$$

from which it follows for $\beta_2$ that

$$\beta_2 = \frac{R_\infty}{R_A}\left(\frac{3}{4} + (2 - \alpha_2 R_A)(R_A - 1)\right).$$

The exponent $\beta_2$ is positive, thus giving

$$\alpha_2 R_A^2 - R_A (2 + \alpha_2) = 5/4 < 0.$$

Subtracting equality (12) from this inequality at $d = 0$, we obtain the inequality

$$R_A > \frac{-1}{4 - 2R_\infty}.$$

which is obvious for $0 < R_\infty < 1$.

Thus, it is proved that $\beta_2 > 0$ when $R_\infty < 1$, i.e., in the case of media with a molecular Prandtl number $\sigma$ higher than unity. The expression for $\beta_2$ can be written in a shorter form as

$$\beta_2 = R_\infty \left(1 - \frac{1}{2R_\infty} + \frac{1}{4R_A}\right). \tag{26}$$

If this value of $\beta_2$ is higher than $\beta_1$ from (24), then branch B1 is realized, since in this case $O(d') < O(K')$. The inequality $\beta_2 \le \beta_1$ or $R_A \le 7R_\infty(12R_\infty - 2)$ serves as a condition for the realization of branch B2. When $\sigma \approx 1$ ($R_\infty \approx 1$), and $R_A \approx 1$ this condition, as is easily seen, is not satisfied. With an increase in the Prandtl number from unity to some transition value $\sigma_{t2} \approx 1.7$, branch B1 is realized, and from $\sigma_{t2}$ to $\sigma = \infty$, branch B2. For branch B2, as for branch B1, $R_A - 1 < 0$, i.e., one should select the root of (12) with the minus sign.

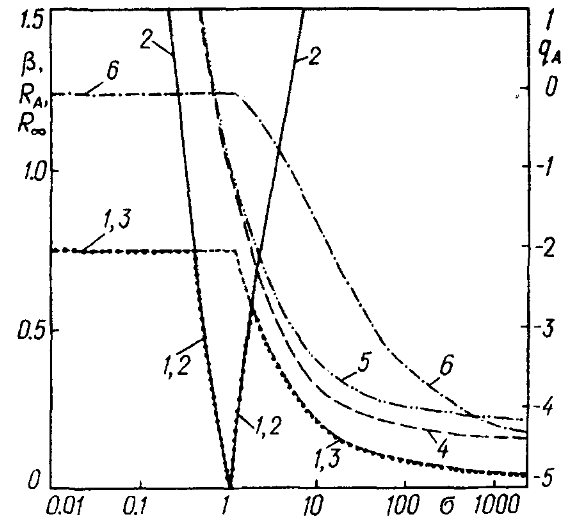


Fig. 1. Dependence of exponents $\beta$, $\beta_1$, $\beta_2$, asymptotic ratios of time scales $R_A$ and $R_\infty$, and mass flow $q_A$ on the molecular Prandtl number: 1) $\beta = \min(\beta_1, \beta_2)$; 2) $\beta_1$; 3) $\beta_2$; 4) $R_A$; 5) $R_\infty$; 6) $q_A$.

Substituting the assumed exponential solutions to system of equations (15), we obtain for the coefficients with these exponents:

in case B1

$$\frac{c_K}{c_q} = -\frac{1}{5(R_A^{-1} - 1)}\left(1 + \frac{2R_\infty}{R_A[(1-\alpha_2 R_A)^{-1} - 2\beta_1]}\right),$$

$$\frac{c_\vartheta}{c_q} = 25\frac{c_K}{c_q}, \quad \frac{c_R}{c_q} = -\frac{5}{2}\frac{R_A R_\infty}{(1-\alpha_2 R_A)^{-1} - 2\beta_1}; \tag{27}$$

in case B2

$$\frac{c_R}{c_q} = \frac{a_4 a_5 - a_3}{a_4 a_5 - a_2 a_3}, \quad \frac{c_d}{c_q} = \frac{a_1 - a_2 a_4}{a_1 a_5 - a_2 a_3},$$

$$\frac{c_K}{c_q} = a_6 \frac{c_d}{c_q}, \quad \frac{c_\vartheta}{c_q} = \left(25 a_6 - \frac{28}{3}\right)\frac{c_d}{c_q}, \tag{28}$$

where

$$a_1 = -3 + \frac{15}{4R_A} - \frac{1}{2R_\infty}, \quad a_2 = \frac{4}{5R_A^2}, \quad a_3 = 1 - \frac{R_A}{R_\infty}, \quad a_4 = -\frac{5}{2}(1 - \alpha_2 R_A) R_A,$$

$$a_5 = \left(-1 + 1/R_\infty - \frac{3}{2R_A}\right)\left(a_6 - \frac{28}{75}\right), \quad a_6 = -98/(-72 + 42R_A^{-1} + 12R_\infty^{-1}).$$

We construct a graph of the dependence of the asymptotic degree of convergence on zero of the corrections on the molecular Prandtl number. For both cases of A2 (versions A2a and A2b) and of B (versions B1 and B2), which describe, correspondingly, Prandtl numbers $\sigma < 1$ and $\sigma > 1$, the exponent $\beta_A$ is determined as $\beta_A = \min(\beta_1, \beta_2)$, where $\beta_1$ depends on the rate of change of function $K'$, and $\beta_2$, on the rate of change of function $d'$.

Figure 1 presents curves of the exponents $\beta_1$ and $\beta_2$ and the resulting exponent $\beta$ versus the Prandtl number. When $\sigma \to 1$ and $\sigma \to \infty$ the exponent $\beta \to 0$. Here the change in the ratio of time scales with the Prandtl

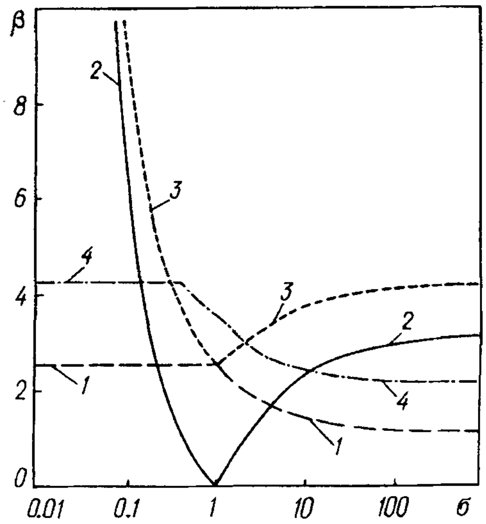


Fig. 2. Dependence of exponents $\beta$ on the molecular Prandtl number: 1) $\beta_E$; 2) $\beta_{\tilde{Q}'/E}$; 3) $\beta_{\tilde{Q}'}$; 4) $\beta_Q$.

number is also shown for the isotropic case $R_\infty(\sigma)$ (formula (4)) and with allowance for stratification $R_A(\sigma)$ (relations (10) and (25)). The difference of the ratio $R_A$ from the isotropic $R_\infty$ starts to manifest itself as $\sigma$ increases from $\sigma = 1$. When $\sigma \to \infty$ the value of $R_A$ tends to about 0.15 and $R_\infty$ to 0.20, so that the ratio $R_A/R_\infty \to 3/4$. Figure 1 also presents the change in the asymptotic value averaged over the oscillations of the transverse mass flow $q_A$ (relations (10) and (11)). When $\sigma > 1$ a turbulent mass flow $q_A$ is formed which differs from zero and which for $\sigma \to \infty$ attains a value of about $-68/15$. The fact that for $\sigma < 1$ $q_A = 0$ and for $\sigma > 1$ $q_A \neq 0$ means, according to the definition of the dimensionless flow $q$, that when $\sigma < 1$ the difference in the decay exponents $Q$ and $E$ is less than unity and when $\sigma > 1$ it is equal to unity.

We now consider how the kinetic energy of turbulence and its components decays. For this purpose we substitute the asymptotic values of $R_A$ and $q_A$ into the equation determining the degree of kinetic energy decay from system (1)

$$\frac{tdE}{Edt} = -\frac{2}{p}(R^{-1} + q),$$

which yields

$$\beta_E = -\frac{tdE}{Edt} = \begin{cases} 5/2 & \text{when } \sigma < 1, \\ R_\infty\left(2 + \dfrac{1}{2R_A}\right) & \text{when } \sigma > 1. \end{cases} \tag{29}$$

For an asymptotically large Prandtl number $\sigma >> 1$ the values of $R_\infty$ and $\sigma_\infty$ are equal to 0.2 and 0.164, respectively. The value of $R_*$ calculated from (12) is equal to 0.15 and $R_A/R_\infty \approx 3/4$. The degree of convergence on zero of the kinetic energy of turbulence when $t \to \infty$ and $\sigma >> 1$ is thus equal to $\lim_{\sigma\to\infty} \beta_E \approx 1$ (Eq. (29)), which is considerably smaller than the exponent 5/2 in the first line of (29) when $\sigma < 1$. In Fig. 2 a graph of the exponent of kinetic energy decay $\beta_E$ versus $\sigma$ is constructed. As $\sigma$ changes within the range of from 1 to about $10^3$, $\beta_E$ varies smoothly from 5/2 to unity.

The exponential relation

$$E = E_0 t^{-\beta_E} \tag{30}$$

which corresponds to the expression for the exponent $\beta_E$, is in good aggrement with the numerical calculation [1] (Fig. 3a).

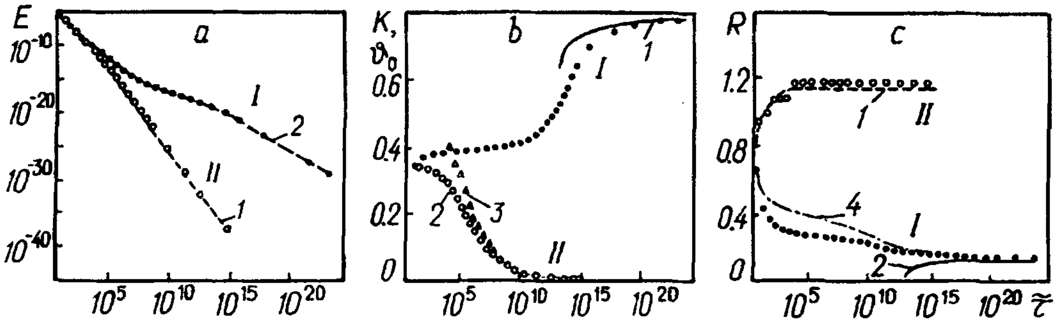

Fig. 3. Comparison of calculation by analytical relations (curves) with the numerical calculation [1] (points): 1) formula (7) for air; 2) (7) for water; 3) $\vartheta$ from [1] for water; 4) formula (25) for water; a) $E(\tilde{\tau})$; b) $K(\tilde{\tau})$; c) $R(\tilde{\tau})$; I) $\sigma = 800$, Fr $= 3.67 \cdot 10^{-2}$; II) $\sigma = 0.73$, Fr $= 2.64 \cdot 10^{-2}$.

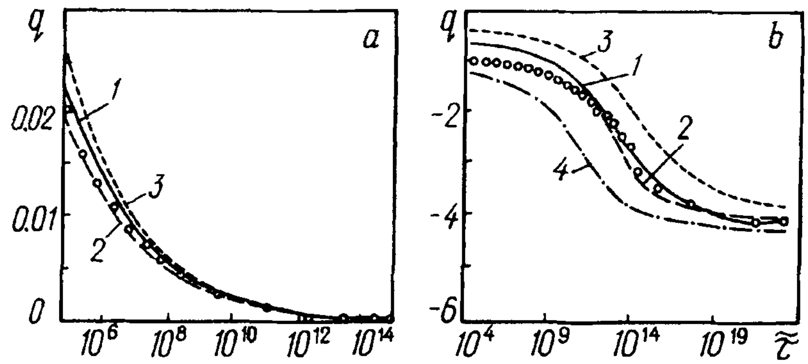

Fig. 4. Comparison of calculation by analytical relations (curves) with the numerical calculation [1] (points): 1) formula (15) from [3]; 2) exponential relation (7); 3) formula (31); 4) formula (11); a) for air ($\sigma = 0.73$; Fr $= 2.64 \cdot 10^{-2}$); b) for water (($\sigma = 800$; Fr $= 3.67 \cdot 10^{-2}$).

When $\sigma > 1$ the components of the kinetic energy of turbulence $R_{11}$, $R_{22}$, and $R_{33}$ decay at the same rate as does $E$. At the initial stage of decay the kinetic energy of turbulence $E$ is mainly determined by its vertical component $R_{22}$. Pumping of the kinetic energy of turbulence from the potential energy of the scalar turbulent field is first performed for the component $R_{22}$, from which it is then redistributed to the other components.

When $\sigma < 1$, $E$ and $R_{22}$ decay at different rates. The degree of convergence on zero of the component $R_{22}$ is $\beta_{R_{22}} = \beta_E + \beta_K$, where $\beta_E$ is found from (29), $\beta_K = \min\,(O(K'), 3/4)$, and $O(K')$ is given by (18). Since $\beta_K > 0$, the component of the energy of turbulent oscillations $R_{22}$ decays more rapidly than the turbulent energy $E$. The field of oscillations becomes, accordingly, plane when $\sigma < 1$ at the final stage of decay and consists of oscillations towards orts 1 and 3.

We analyze various analytical relations for $q$. Differentiating the third equation in (8) and then developing the values of the derivatives from system (6), we obtain for the flow $q$

$$q = -\frac{(1/R - 1)(K - 1)\vartheta}{(1 + \vartheta)(2K - 9/5)},$$

which is a particular case of (16) from [3] for $d \to 1$ and $p_1 \to 0$. Employing the fact that at $d = 1$ from $A_1 = 0$ there follows the relation $K = \vartheta/(\vartheta + 1)$, we eliminate $\vartheta$ from the latter relation

$$q = -\frac{(1/R - 1)(K - 1)K}{2K - 9/5}. \tag{31}$$

Just as expression (15) in [3], relation (31) is universal with respect to the Prandtl number and is fulfilled satisfactorily (Fig. 4a) for air not only in the final, but also within the entire far field. For water (Fig. 4b) this correspondence is also present, but it is poorer, due to the much slower convergence of the parameter $d$ on unity. This is proved by a comparison of relation (31) with the more general expression (15) from [3] for $q$ (Figs. 4a and 4b). We recall here that for air with $\sigma = 0.73$ an asymptotic version A2b is realized, where the function $d'$ decays more rapidly than the others in system of equations (6), and for water with $\sigma = 800$ it decays at the same rate as the other functions converge on their asymptotic values.

Agreement of the exponential solution in the case of water with the numerical calculation exists only when $\tilde{\tau} > 10^{11}$ (see Fig. 4b). At the same time, functional relation (11) is approximately fulfilled for the entire far field. For air, the exponential solution is in rather good agreement with the numerical solution. The coefficient $c_q$ in exponential relation (7) is found from the numerical solution when $t \to \infty$. Having determined $c_q$, one can calculate (using (21) for air and (28) for water) the coefficients of the exponential relations and the other considered functions and compare them with their numerical solutions.

In Fig. 3b, this comparison is performed for the portion $K$ introduced by transverse oscillations to the kinetic energy of turbulence. For air, the region of correspondence of the exponential solution (7) to the exponent (19) and the coefficient of exponent (21) to the numerical solution [1] is wider than for water with the solution determined by formulas (7), (26), (28). In the case of air, the exponential relations at the final stage coincide for the functions $K$ and $\vartheta$; therefore, Fig. 3b also presents the data of the numerical solution for the function $\vartheta$. In this case, rather good agreement is also noted.

The ratio $R$ for air calculated by the exponential law (Fig. 3b) is practically in precise agreement with the numerical solution. For water, the agreement with the exponential solution begins only after $\tilde{\tau} = 10^{15}$. Formula (25) for the asymptotic limit of $R_A$ has a wider applicability range.

Having found the asymptotic values of all the functions, one can calculate the frequency of oscillations and the decay rate of their amplitudes in the final stage. For this purpose, the results of [3] are used. There are no oscillations in degenerate system of equations (6). It describes the behavior of functions averaged over internal gravitation waves (see [3]).

Thus, for angular frequency $\omega$, from relation (19) in [3] when $d \to 1$ we have

$$\omega^2 = (2\,(1 + \vartheta)\left(\frac{9}{5} - 2K\right) . \tag{32}$$

Formula (32) can be even more simplified by allowing for the fact that in the final stage of decay it follows from $A_1 = 0$ that $1 + \vartheta = (1 - K)^{-1}$, from which we obtain the dependence of $\omega$ only on $K$

$$\omega^2 = \frac{2}{1 - K}\left(\frac{9}{5} - 2K\right) , \tag{33}$$

which in the case of a medium with $\sigma < 1$ leads to the asymptotic value $\omega^2 = 18/5$, and with $\sigma > 1$ to $\omega^2 = 2$. In accordance with this, the period of oscillations $T = 2\pi/\omega$ in the asymptotics is equal to $T = \sqrt{10}\,\pi/3 = 3.30$ for $\sigma < 1$ and to $T = \sqrt{2}\pi = 4.44$ for $\sigma > 1$.

The decay rate of the amplitude of mass flow oscillations $\tilde{q}'$ in the general case is dictated by expression (27); assuming $d' = 0$ in this expression at the final stage of decay we write for the amplitude exponent

$$\beta_{\tilde{q}'} = -\frac{t d\tilde{q}'}{\tilde{q}' dt} = -\left[\frac{5}{2}\left(\frac{R_\infty}{R_A} - \sigma_\infty - \frac{3}{5}\right) + 5 q_A R_\infty\right] - 1 . \tag{34}$$

Analysis of expression (34) shows that in the vicinity of $\sigma = 1$ the exponent $\beta_{\tilde{q}'}$ is negative and reduces to $-1$ at $\sigma = 1$. The negative character of $\beta_{\tilde{q}'}$ does not contradict the physical meaning, though it means an unlimited growth of oscillation amplitude for $\tilde{q}$. Thus, the same exponent for functions $Q$ and $Q/E$ is positive in all cases. Since oscillations of $\tilde{E}$ are small compared to the mean value of $E$, we write approximately $\tilde{q} = \tilde{Q}t/E$, from which it follows that

$$\beta_{\tilde{q}'} = -\frac{td\tilde{q}'}{\tilde{q}'dt} = -\frac{td\tilde{Q}'}{\tilde{Q}'dt} + \frac{tdE}{Edt} - 1,$$

$$\beta_{\tilde{Q}'} = \beta_{\tilde{q}'} + \beta_E + 1, \quad \beta_{\tilde{Q}'/E} = \beta_{\tilde{q}'} + 1. \tag{35}$$

In Fig. 2 the dependences, calculated from (34), (35), and (29), of the degrees of the decay rate of the amplitude of oscillation of the corresponding functions on the molecular Prandtl number $\sigma$ are also constructed. This figure also presents the calculated exponent for $Q$, which is equal to $\beta_Q = \beta + \beta_E + 1$ when $\sigma < 1$ and $\beta_Q = \beta_E + 1$ when $\sigma > 1$. The amplitude of oscillations of turbulent mass flow $\tilde{Q}$ decays more rapidly than the mean value of $Q$ in the case of water with $\sigma = 800$, and more slowly than $Q$ for air with $\sigma = 0.73$. Consequently, at the final stage of decay on the graph of the simusoidal dependence $Q(\tau)$ there should exist a component which is noticeable compared to the amplitude and which changes slowly and therefore resembles a constant. For air, this component should be practically insignificant. In numerical calculation, the opposite situation is observed (Fig. 22 from [1 ]).

The convergence noted in [1 ] of the ratio of kinetic energy of velocity vertical fluctuations $K$ to the potential energy of the density field $\vartheta$ on unity (as in the given analysis) is a confirmation of the fact that in the case of air with $\sigma < 1$ the mean value of $q$ decays more rapidly than $\tilde{q}$ for $\tau \to \infty$. This means a complete transition of turbulence to internal waves. In the case of water with $\sigma > 1$ the calculated value of the limit is $\lim_{\tau \to \infty} K/\vartheta = 1/5$, and this must correspond to the presence of a constant component in the graph $Q(\tau)$.

We now direct our attention to the case set that was denoted as A1 and corresponds to the Prandtl number $\sigma = 1$. Here $q_A = 0$, $R_A = R_\infty = 1$, $\vartheta_A = K_A/(1 - K_A)$, and the value of $K_A$ is undetermined. The analysis of this degenerate case is somewhat more cumbersome than that of cases A2 and B. The asymptotic quantity $K_A$ is, for example, capable of taking the values 0, 1/3 and 9/10. Other versions are possible with the exponent $\beta$ varying with $K_A$. A selection among all these versions can be made by the criterion of continuity of $\beta$ variation with the Prandtl number. It is seen from Fig. 2 that a zero value of $\beta$ is natural for $\sigma = 1$. This corresponds to the degenerate solution of system (35)

$$\beta_K = \beta_R = \beta_\vartheta = \beta_q = 0, \quad \beta_d = 3/4, \quad q' = R' = \Theta' = K' = 0$$

$$K_A = 1/3, \quad q_A = 0, \quad \vartheta_A = 1/2, \quad R_A = 1, \tag{36}$$

for which total isotropy of turbulence is the final state.

**Discussion and conclusions.** In the analysis of the final stage of turbulence decay we confirmed analytically the convergence of the functions $K$ and $R$ on different limits for $\sigma < 1$ and $\sigma > 1$, which was previously noted in [1 ], and calculated these limits as functions of the molecular Prandtl number.

Three cases are distinguished, A1, A2, and B, which differ in the asymptotic values of the ratio $K$, 1/3, 0, and 4/5, and of all other functions. These asymptotic values are given by expressions (9)-(12). The rate of convergence of the solution on these asymptotes in the form of (7) is studied. On the basis of this analysis we concluded that case A2 corresponds to $\sigma < 1$, case B to $\sigma > 1$, and case A1 to $\sigma = 1$. Cases A2 and B split into two asymptotic branches each: A2a, A2b and B1, B2 with different rates of convergence on the asymptotic limits. The transition from branch A2a to branch A2b takes place at $\sigma_{t_1} \simeq 0.13$, as the Prandtl number increases from zero to unity. The rate of convergence for range A2a is equal to $-3/4$ for all the functions, and the coefficients of the exponents in (7) are given by expressions (16)-(17). For branch A2b, these values are calculated from formulas (18) and (21), respectively.

The transition from branch B1 to branch B2 of case B occurs at $\sigma_{t_2} \simeq 1.7$, as the Prandtl number increases from unity to $\sigma >> 1$. The exponent in (7) for these two branches is calculated based on (24) and (26), and the coefficients of the exponents, from (27) and (28), respectively.

The kinetic energy of turbulence in the case of density-stratified media with molecular Prandtl numbers $\sigma < 1$ and $\sigma > 1$ decays according to different exponential laws. The rate of decay is determined for $\tau \to \infty$ from

(29). When $\sigma > 1$ it decays much more slowly than in the isotropic case, due to the fact that a great portion of this energy is confined in regular oscillations with weak decay.

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## NOTATION

$\tau$, dimensionless time; $\rho$, density; $U$, flow velocity; $M$, grid cell size; $\varepsilon = \mathrm{Fr}^2$; $\mathrm{Fr} = N_{BV}M/U$, Froude number; $N_{BV} = [(gd\bar{\rho})/(\bar{\rho}dx_2)]^{1/2}$, Brunt-Väisälä number; $\tilde{\tau} = \mathrm{Fr}\cdot\tau$; $T_u$, dimensionless time scale of the velocity field; $T_\rho$, dimensionless time scale of the density field; $t = \varepsilon T_\rho$; $R = T_u/T_\rho$, ratio of time scales of velocity field and scalar field; $E$, dimensionless kinetic energy; $R_{22}$, vertical component of the tensor of velocity fluctuations; $K = R_{22}/E$, portion of the energy of transverse oscillations in the kinetic energy of turbulence; $Q$, dimensionless turbulent transverse mass flow; $q = \varepsilon T_\rho Q/E$; $\Theta$, dimensionless square of density fluctuations; $\vartheta = \varepsilon\Theta/E$; $R_\lambda = (5ET_u\mathrm{Re})^{1/2}$, turbulent Reynolds number; $\mathrm{Re} = UM/\nu$, Reynolds number; $\sigma$, molecular Prandtl number.

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